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Limit Brody curves and lifting map problems from symmetrized polydisc in small dimensions

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Introduction

This thesis consists of three main chapters with the essential content as follows:

Chapter 1 presents a generalization of the result of Do Duc Thai, Mai Anh Duc and Ninh Van Thu on classes of manifolds which are not of $E-$limit type, or also called manifolds which do not contain limit $E-$Brody curves. This work is published in [3].

Chapter 2 presents results on the lifting map problem from symmetrized polydisc $\mathbb{G}_n$ to spectral ball Ω_n with $n \leq 5$, where lifting maps are made precise and we prove that the lifting map formula does not work for dimensions $n \geq 6$. This work is published in [1].

Chapter 3 presents results on the lifting map problem from the symmetrized polydisc $\mathbb{G}_4$ to the spectral ball Ω_4 with given first derivative. This work is a continuation of the one of N. Nikolov, P. Pflug and P. J. Thomas and more precisely, we point out the nature of the necessary and sufficient conditions for local liftability. This work is new, still in form of a preprint and has not been published in any journal yet.

Chapter 1

On limit $E-$Brody curves

1.1 Introduction

In this chapter, we present a new proof of a result of Do Duc Thai, Mai Anh Duc and Ninh Van Thu concerning limit Brody curves and non-normal families of holomorphic mappings. Non-normal families of holomorphic mappings and Brody curves (i.e. a holomorphic curve with bounded derivatives) are closely related. Concerning these topics, these three authors proved the following results.

Theorem 1.1. $\mathbb{C}^n$ $(n \geq 2)$ *is not of E-limit type for any length function E on $\mathbb{C}^n$.*

Theorem 1.2. $(\mathbb{C}^*)^2$ *is not of ds^2_{FS}-limit type, where ds^2_{FS} is the Fubini-Study metric on $\mathbb{P}^2(\mathbb{C})$.*

Their proof makes use of a result of J. Winkelmann and some quite complicated techniques of extraction of subsequences. So in this note, we try to give another short and slightly more general proof.

First of all, we recall some definitions to explain the content of their theorems.

Definition 1.3. Let X be a complex manifold with a hermitian metric E. A holomorphic curve $f : \mathbb{C} \to X$ is said to be an **E-Brody curve** if its derivative is bounded, i.e., $|f'(z)|_E \leq c$ for every $z \in \mathbb{C}$ where c is a constant positive constant.

Length function is a more general notion of metric, but it will not concern us in this note, so we refer the reader to their paper for the definition. Note that if we write $E_p(\vec{v})$, it means the length of the tangent vector $\vec{v}$ at the point p with respect to the metric E. If the point p is well understood, $|\vec{v}|_E$ is sufficient.

Definition 1.4. Let X be a complex manifold with a hermitian metric E. The complex manifold X is said to be of **E-limit type** if X satisfies the following:

For each non-normal family $\mathcal{F} \subset \mathrm{Hol}(\Delta, X)$, where Δ is a domain in $\mathbb{C}$ and $\mathrm{Hol}(\Delta, X)$ the set of all holomorphic mappings from Δ into X, such that $\mathcal{F}$ contains no compactly divergent sequence, then there exist sequences $\{p_j\} \subset \Delta$ with $p_j \to p_0 \in \Delta$ as $j \to \infty$, $\{f_j\} \subset \mathcal{F}, \{\rho_j\} \subset \mathbb{R}$ with $\rho_j > 0$ and $\rho_j \to 0^+$ as $j \to \infty$ such that

$$g_j(\xi) := f_j(p_j + \rho_j \xi), \xi \in \mathbb{C},$$

converges uniformly on any compact subsets of $\mathbb{C}$ to a non-constant E-Brody curve $g : \mathbb{C} \to X$.

For the convenience of the reader, we recall the definition of a normal family and compact divergence.

Definition 1.5. A family $\mathcal{F} \subset \mathrm{Hol}(\Delta, X)$ is said to be **normal** if, for each sequence $\{f_j\}_{j=1}^{\infty}$ in $\mathcal{F}$, there exists a subsequence which converges uniformly on compact subsets of Δ. A family of mappings which is not normal is called a **non-normal family**.

Definition 1.6. A sequence $\{f_j\}_{j=1}^{\infty}$ in $\mathrm{Hol}(\Delta, X)$ is said to be **compactly divergent** if, for all compacts $K \subset \Delta$ and $L \subset X$, there exists j_0 such that, for $j \geq j_0$, we have $f_j(K) \cap L = \emptyset$.

1.2 Non-existence of limit Brody curves

Now we present our main result which is a slight generalization of their Theorem 1.2 stated as follows.

Theorem 1.7 (Main result)**.** *Let X be a complex manifold which contains an entire curve, i.e. there exists a non-constant holomorphic curve $f : \mathbb{C} \to X$. Then both $\mathbb{C} \times X$ and $\mathbb{C}^* \times X$ are not of $E-$limit type for any hermitian metric E on $\mathbb{C} \times X$ or $\mathbb{C}^* \times X$ respectively.*

In the proof of the theorem, we make use of two lemmas.

Lemma 1.8. *Let $c_n > 0$ for $n \in \mathbb{N}$. The following are equivalent.*
(a) $\prod_{n=1}^{\infty}(1 + c_n) < \infty$, (b) $\sum_{n=1}^{\infty} c_n < \infty$.
Moreover, (a) and (b) are equivalent to the following
(c) $\prod_{n=1}^{\infty}(1 - c_n) > 0$ if we suppose in addition $0 < c_n < 1$ for every n.

The second is the main lemma, which is the key of the proof of the Main result.

Lemma 1.9. *Let* $\{\alpha_j\}_{j=1}^{\infty}$ *be a sequence of pairwise distinct nonzero complex numbers such that* $\sum_{j=1}^{\infty} \frac{1}{|\alpha_j|} < \infty$. *Then, for all complex numbers* p_j *and* k_j *with* $1 \leq j \in \mathbb{N}$, *there exists a holomorphic function* $g\colon \mathbb{C} \to \mathbb{C}$ *which satisfies the following interpolation conditions:* $g(\alpha_j) = p_j$ *and* $g'(\alpha_j) = k_j$ *for every* $1 \leq j \in \mathbb{N}$.

To prove this lemma we make use of a basic technique in sheaf theory.

Chapter 2

Lifting map problem from symmetrized polydisc without derivative conditions

2.1 Résumé

The spectral unit ball Ω_n is the set of all $n \times n$ matrices M with spectral radius less than 1. Let $\pi(M) \in \mathbb{C}^n$ stand for the coefficients of the characteristic polynomial of a matrix M (up to signs), i.e. the elementary symmetric functions of its eigenvalues. The symmetrized polydisc is $\mathbb{G}_n := \pi(\Omega_n)$.

When investigating Nevanlinna-Pick problems for maps from the disk to the spectral ball, it is often useful to project the map to the symmetrized polydisc (for instance to obtain continuity results for the Lempert function): if $\Phi \in \mathrm{Hol}(\mathbb{D}, \Omega_n)$, then $\pi \circ \Phi \in \mathrm{Hol}(\mathbb{D}, \mathbb{G}_n)$. Given a map $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{G}_n)$, we are looking for necessary and sufficient conditions for this map to "lift through given matrices", i.e. find Φ as above so that $\pi \circ \Phi = \varphi$ and $\Phi(\alpha_j) = A_j$, $1 \leq j \leq N$. A natural necessary condition is $\varphi(\alpha_j) = \pi(A_j)$, $1 \leq j \leq N$. When the matrices A_j are derogatory (i.e. do not admit a cyclic vector) new necessary conditions appear, involving derivatives of φ at the points α_j. We prove that those conditions are necessary and sufficient for a local lifting. We give a formula which performs the global lifting in small dimensions ($n \leq 5$), and a counter-example to show that the formula fails in dimensions 6 and above.

2.2 Introduction

Some problems in Robust Control Theory lead to the study of structured singular values of a matrix (denoted by μ). A special case of this is simply the spectral radius. A very special instance of the "μ-synthesis" problem reduces to a Nevanlinna-Pick problem, i.e. given points $\alpha_j \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $A_j \in \Omega \subset \mathbb{C}^m$, $1 \leq j \leq N$, determine whether there exists Φ holomorphic from $\mathbb{D}$ to Ω such that $\Phi(\alpha_j) = A_j$, $1 \leq j \leq N$. We refer the interested reader to Nicholas Young's stimulating survey.

We study this special case. Let us set some notation.

Let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices. For $A \in \mathcal{M}_n$ denote by $\mathrm{Sp}(A)$ and $r(A) = \max_{\lambda \in \mathrm{Sp}(A)} |\lambda|$ the spectrum and the spectral radius of A, respectively.

Definition 2.1. The *spectral ball* Ω_n is given as

$$\Omega_n := \{A \in \mathcal{M}_n : r(A) < 1\}.$$

The *symmetrized polydisc* $\mathbb{G}_n$ is defined by

$$\mathbb{G}_n := \{\pi(A) : A \in \Omega_n\},$$

where the mapping $\pi : \mathcal{M}_n \longrightarrow \mathbb{C}^n$, $\pi = (\sigma_1, \ldots, \sigma_n)$, is given, up to alternating signs, by the coefficients of the characteristic polynomial of the matrix:

$$P_A(t) := \det(tI_n - A) =: \sum_{j=0}^{n} (-1)^j \sigma_j(A) t^{n-j}.$$

In other words, the k-th coordinate of π, $\sigma_k(A)$, is the k-th elementary symmetric function of the eigenvalues of A.

Problem 2.2. *(The Lifting Problem).*

Given a map $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{G}_n)$ and $A_1, \ldots, A_N \in \Omega_n$, find conditions (necessary, or sufficient) such that there exists a $\Phi \in \mathrm{Hol}(\mathbb{D}, \Omega_n)$ satisfying $\varphi = \pi \circ \Phi$ and $\Phi(\alpha_j) = A_j$ for $j = 1, \ldots, N$.

When this happens, we say that the map φ *lifts* through the matrices $A_1, \ldots, A_N$ at $(\alpha_1, \ldots, \alpha_N)$. An obvious necessary condition for φ to lift through the matrices $A_1, \ldots, A_N$ at $(\alpha_1, \ldots, \alpha_N)$ is that $\varphi(\alpha_j) = \pi(A_j)$ for $j = 1, \ldots, N$.

Remark 2.3. Whenever there is a solution to the Lifting Problem for α_j, A_j, then there is one for $\alpha_j, \tilde{A}_j$, when $A_j \sim \tilde{A}_j$ for each j, i.e. A_j is similar to $\tilde{A}_j$, i.e. for each j there exists $P_j \in \mathcal{M}_n^{-1}$ such that $\tilde{A}_j = P_j^{-1} A_j P_j$.

2.3 First reductions of the problem

We need to establish some notations.

Definition 2.4. Given a vector $v := (v_1, \dots, v_n) \in \mathbb{C}^n$, we denote $P_{[v]}(t) := t^n + \sum_{j=1}^n (-1)^j v_j t^{n-j}$.

This choice ensures that $P_{[\pi(A)]} = P_A$.

Definition 2.5. Given $a := (a_1, \dots, a_n) \in \mathbb{C}^n$, the *companion matrix* of a is

$$C_{[a]} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix}.$$

We see that its characteristic polynomial is then $\det(tI_n - C_{[a]}) = t^n - \sum_{j=1}^n a_j t^{n-j}$, so that $\sigma_j(C_{[a]}) = (-1)^{j+1} a_j$, for $1 \le j \le n$.

Given a matrix M, the *companion matrix* of M, denoted C_M, is the unique matrix in companion form with the same characteristic polynomial as M.

A matrix $A \in \mathcal{M}_n$ is cyclic (i.e. non-derogatory) if and only if it is conjugate to its companion matrix.

The above computation of the characteristic polynomial of a companion matrix shows that $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{G}_n)$, if we write $\tilde{\varphi} := ((-1)^{j+1}\varphi_j, 1 \le j \le n)$, then the map given by $\Phi(\zeta) := C_{[\tilde{\varphi}(\zeta)]}$ is a lifting of φ.

Therefore, in view of Remark 2.3, this means that lifting through a set of cyclic matrices can be achieved as soon as the obvious necessary conditions $\varphi(\alpha_j) = \pi(A_j)$, $1 \le j \le N$, are satisfied.

The case where A_1 has only one eigenvalue, and $A_2, \dots, A_N$ are cyclic, has been studied in the article of P. J. Thomas and Nguyen Van Trao.

2.4 Necessary conditions

Let $A_1 \in \mathcal{M}_n$. Up to conjugacy, we may assume that it is in Jordan form. Write this in blocks associated to each of the *distinct* eigenvalues of A_1, denoted $\lambda_k, 1 \le k \le s$ where $s \le n$. Namely

$$A_1 = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_s \end{pmatrix}, \quad B_k \in \mathcal{M}_{m_k}, \quad \sum_{k=1}^s m_k = n, \tag{2.1}$$

where Sp $B_k = \{\lambda_k\}$, and $\lambda_j \neq \lambda_k$ for $k \neq j$.

Temporarily, we fix k and write (B, λ, m) instead of (B_k, λ_k, m_k). We need to set up some notation as in the article of P. J. Thomas and Nguyen Van Trao. Let $B = (b_{i,j})_{1 \leq i,j \leq m}$. Then $b_{jj} = \lambda$, $b_{j-1,j} \in \{0, 1\}$, $2 \leq j \leq m$, and $b_{ij} = 0$ if either $i > j$ or $i + 1 < j$.

Let r stand for the rank of $B - \lambda I_m$, so there are exactly exactly $m - r$ columns in $B - \lambda I_m$ which are identically zero, the first, and the ones indexed by the integers $j \geq 2$ such that $b_{j-1,j} = 0$. Enumerate the (possibly empty) set of column indices where the coefficient $b_{j-1,j}$ vanishes as

$$\{j : b_{j-1,j} = 0\} =: \{b_2, \ldots, b_{m-r}\}, 2 \leq b_2 < \cdots < b_{m-r} \leq m.$$

Equivalently, $b_{l+1} - b_l$ is the size of the Jordan block $B^{(l)} := (b_{ij})_{b_l \leq i,j \leq b_{l+1}-1}$. The integer $b_{l+1} - b_l$ is also the order of nilpotence of the block $B^{(l)}$.

We choose the Jordan form so that $b_{l+1} - b_l$ is increasing for $1 \leq l \leq m - r$, with the convention $b_{m-r+1} := m+1$. It means that the possible zeroes appear for the smallest possible indices j, globally.

Definition 2.6. For $1 \leq i \leq m$,

$$d_i(B) = d_i := 1 + \#\{k : m - i + 2 \leq b_k \leq m\}.$$

Equivalently, $d_i - 1$ is the number of columns which are identically zero, among the last $i - 1$ columns of A_1, or

$$d_i = 1 + (m - r) - \max\{j : b_j \leq m - i + 1\},$$

with the agreement that the maximum equals 0 if the set on the right hand side is empty.

One can also interpret $d_j = d_j(B)$ as the least integer d such that there is a set S of d vectors in $\mathbb{C}^n$ with the property that the the iterates of S by B span a subspace of $\mathbb{C}^n$ of dimension at least j (we shall not need this characterization, so we do not include a proof).

Notice that B is cyclic if and only if $d_i(B) = 1$, for any i ($b_{j-1,j} = 1$ for any j); while it is scalar if and only if $d_i(B) = i$, for any i ($b_{j-1,j} = 0$ for any j).

The following proposition gives a set of conditions for lifting which are locally necessary and sufficient. This says in particular that all possible necessary conditions that can be obtained from the behavior of Φ in a neighborhood of $\alpha \in \mathbb{D}$ are exhausted by (3.8).

Proposition 2.7. *Let $\varphi \in Hol(\omega, \mathbb{G}_n)$, where ω is a neighborhood of $\alpha \in \mathbb{D}$. Let A_1 be as in (2.1). Then the following assertions are equivalent:*

(a) There exists $\omega' \subset \mathbb{D}$ a neighborhood of α and $\Phi \in \mathrm{Hol}(\omega', \Omega_n)$ such that

$$\pi \circ \Phi = \varphi, \Phi(0) = A_1;$$

(b) The map φ verifies

$$\frac{d^k P_{[\varphi(\zeta)]}}{dt^k}(\lambda_j) = O((\zeta - \alpha)^{d_{m_j - k}(B_j)}), \quad 0 \leq k \leq m_j - 1, 1 \leq j \leq s, \quad (2.2)$$

where the d_i are as in Definition 2.6.

(c) There exists $\omega' \subset \mathbb{D}$ a neighborhood of α and $\Phi \in Hol(\omega', \Omega_n)$ such that

$$\pi \circ \Phi = \varphi, \Phi(0) = A_1 \text{ and } \Phi(\zeta) \text{ is cyclic for } \zeta \in \omega' \setminus \{\alpha\}.$$

2.5 Main results

First we need a linear algebra lemma giving us a canonical form for matrices, slightly different from the Jordan form and adapted to our purposes. We will call it *modified Jordan form.*

We need to set some slightly modified notations. Let $A_1 \in \mathcal{M}_n$, with $Sp(A_1) = \{\lambda_1, \ldots, \lambda_n\}$: here the eigenvalues are repeated according to their multiplicities. Let $m_1, \ldots, m_s$ be the respective multiplicities, set $n_j = m_1 + \cdots + m_j$. Therefore $0 = n_0 < n_1 < \cdots < n_s = n$ and $\lambda_k = \lambda_{k'}$ if and only if there exists $i \in \{1, \ldots, s\}$ such that $n_{i-1} < k, k' \leq n_i$.

We assume that $A_1 := (a_{ij})_{1 \leq i,j \leq n}$ is in Jordan form with the notations of (2.1), except for the labeling of the eigenvalues: now Sp $B_k = \{\lambda_{n_k}\}$.

Lemma 2.8. *A matrix A_1 as given above is conjugate to $A' = (a'_{ij})_{1 \leq i,j \leq n}$ where $a'_{n_i, 1+n_i} = 1 \neq a_{n_i, 1+n_i} = 0$, $1 \leq i \leq s-1$, and $a'_{ij} = a_{ij}$ for all other values of the indices.*

Proposition 2.9. *Let Φ be the map from $\mathbb{D}$ (except for some singularities) to $\mathcal{M}_n$ defined by*

$$\Phi(\zeta) := \begin{pmatrix} \varphi_{1,1}(\zeta) & f_2(\zeta) & 0 & \cdots & 0 \\ 0 & \varphi_{2,2}(\zeta) & \ddots & & \vdots \\ \vdots & & \ddots & f_{n-1}(\zeta) & 0 \\ 0 & 0 & \cdots & \varphi_{n-1,n-1}(\zeta) & f_n(\zeta) \\ \varphi_{n,1} & \varphi_{n,2} & \cdots & \varphi_{n,n-1} & \varphi_{n,n} \end{pmatrix},$$

where the f_k, $2 \le k \le n$, and $\varphi_{k,k}$, $1 \le k \le n-1$, are holomorphic functions to be chosen, the f_k are not identically zero, and where

$$\varphi_{n,\ell} := -\frac{\Delta^{\ell-1} P_{[\varphi(\zeta)]}(\varphi_{1,1}, \dots \varphi_{\ell,\ell})}{\prod_{k=\ell+1}^{n} f_k(\zeta)}, \quad 1 \le \ell \le n-1, \tag{2.3}$$

and finally

$$\varphi_{n,n} := -\Delta^{n-1} P_{[\varphi(\zeta)]}(\varphi_{1,1}, \dots, \varphi_{n-1,n-1}, 0) = \varphi_1 - (\varphi_{1,1} + \cdots + \varphi_{n-1,n-1}).$$

Then $\pi \circ \Phi(\zeta) = \varphi(\zeta)$, for the values of ζ where the quotients make sense.

Let A' be as in the conclusion of Lemma 2.8, and $\alpha \in \mathbb{D}$. If $f_k(\alpha) = a'_{k-1,k}$, $2 \le k \le n$, $\varphi_{kk}(\alpha) = \lambda_k$, $1 \le k \le n$, and $\varphi_{n,k}(\alpha) = 0$, $1 \le k \le n-1$, then $\Phi(\alpha) = A'$.

If $f_k(\zeta) \neq 0$ for any $k \in \{2, \dots, n\}$ and $\zeta \in \mathbb{D} \setminus \{\alpha\}$, then $\Phi(\zeta)$ is cyclic for $\zeta \in \mathbb{D} \setminus \{\alpha\}$.

The most important result of the chapter is the following:

Theorem 2.10. *Let $n \in \mathbb{N}^*$, $n \le 5$, $A_1, \dots, A_N \in \mathcal{M}_n$, $\alpha_1, \dots, \alpha_N \in \mathbb{D}$ and $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{G}_n)$.*

Then there exists $\Phi \in \mathrm{Hol}(\mathbb{D}, \Omega_n)$ satisfying $\varphi = \pi \circ \Phi$ and $\Phi(\alpha_j) = A_j$ for $j = 1, \dots, N$ if and only if φ satisfies the conditions (3.8) for each j, with A_j instead of A_1 and α_j instead of α, for $1 \le j \le N$.

Furthermore, the values $\Phi(\zeta)$ may be chosen as cyclic matrices when $\zeta \notin \{\alpha_1, \dots, \alpha_n\}$.

To prove this result, we make use of the lifting map formula above and combine computation techniques and we give proofs for each case of Jordan form of interpolation points.

Chapter 3

Lifting map problem from symmetrized polydisc with first derivative condition

3.1 Statement of the problem

Recall that $\pi\colon \Omega_4 \to \mathbb{G}_4$ is the usual projection (or symmetrization mapping) (cf Definition 2.1). Let $B_0 \in \Omega_4$ and B_1 a complex square matrix of size 4. Let $\varphi\colon \mathbb{D} \to \mathbb{G}_4$ be a holomorphic disc such that $\varphi(0) = \pi(B_0)$ and $\varphi'(0) = D\pi_{B_0}B_1$, where $D\pi_{B_0}$ is the derivative of π at B_0. Such a triplet is called a datum, denoted by $\{\varphi, B_0, B_1\}$.

We say the datum $\{\varphi, B_0, B_1\}$ is **locally liftable** if there exists a holomorphic mapping $\Phi\colon \omega \to \Omega_4$ where ω is a neighbourhood of $0 \in \mathbb{C}$ such that

$$\begin{cases} \varphi = \pi \circ \Phi \text{ on } \omega \\ \Phi(0) = B_0 \\ \Phi'(0) = B_1 \end{cases} .$$

In this case, we call Φ the (local) lifting map of φ.

Problem 3.1. *Let $\{\varphi, B_0, B_1\}$ be a datum. Find necessary and sufficient conditions on φ so that the datum $\{\varphi, B_0, B_1\}$ is locally liftable.*

3.2 First reductions of the problem

We say that two data $\{\varphi, B_0, B_1\}$ and $\{\varphi, B_0', B_1'\}$ are *equivalent* if there exist holomorphic mappings $\Phi\colon \omega \to \Omega_4$ and $P\colon \omega \to \mathrm{GL}(4, \mathbb{C})$ such that

$\begin{cases} \Phi(0) & = B_0 \\ \Phi'(0) & = B_1 \end{cases}$ and $\begin{cases} (P^{-1}\cdot\Phi\cdot P)(0) & = B_0' \\ (P^{-1}\cdot\Phi\cdot P)'(0) & = B_1' \end{cases}$. Note that φ is not involve in the definition of equivalence. We easily see that if two data are equivalent, then the liftability of one datum implies that of the other.

We look at the relation between (B_0, B_1) and (B_0', B_1') in more detail: Firstly, we have $B_0' = P(0)^{-1}B_0P(0)$, therefore the two matrices B_0 and B_0' are similar. This implies that two data (φ, B_0, B_1) and $(\varphi, C^{-1}B_0C, C^{-1}B_1C)$ are equivalent for every $C \in \mathrm{GL}(n, \mathbb{C})$. So we can suppose B_0 is in Jordan form, and from now on, we will do so.

Now suppose two data (φ, B_0, B_1) and (φ, B_0, B_1') are equivalent. Then $P(0)$ is a matrix commuting with B_0. Moreover,

$$\begin{aligned} B_1' &= P(0)^{-1}B_1P(0) + P(0)^{-1}B_0P'(0) - P(0)^{-1}P'(0)P(0)^{-1}B_0P(0) \\ &= P(0)^{-1}B_1P(0) + P(0)^{-1}B_0P'(0) - P(0)^{-1}P'(0)B_0 \\ &\qquad (\text{ since } P(0) \text{ commutes with } B_0) \\ &= P(0)^{-1}B_1P(0) + P(0)^{-1}\Big(B_0P'(0) - P'(0)B_0\Big) \\ &= P(0)^{-1}B_1P(0) + P(0)^{-1}[B_0, P'(0)] \end{aligned} \tag{3.1}$$

where $[B_0, P'(0)] = B_0P'(0) - P'(0)B_0$ is the usual commutator.

Therefore, we have

Proposition 3.2. *Two data (φ, B_0, B_1) and $(\varphi, B_0, C^{-1}B_1C + C^{-1}[B_0, M])$ are equivalent where $C \in \mathrm{GL}(n, \mathbb{C})$ is commuting with B_0 and $M \in \mathbb{C}^{n,n}$ any square matrix.*

Suppose $\mathrm{Sp}(B_0) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ where the λ_i's are distinct and

$$B_0 = \begin{pmatrix} B_{0,\lambda_1} & & & \\ & B_{0,\lambda_2} & & \\ & & \ddots & \\ & & & B_{0,\lambda_k} \end{pmatrix}$$

is the block form of B_0 where $\mathrm{Sp}(B_{0,\lambda_i}) = \{\lambda_i\}$. Recall the notation $P_{[\varphi(\zeta)]}(t) = \sum_{j=0}^{n}(-1)^j\varphi_j(\zeta)t^{n-j}$ where $\varphi_0 \equiv 1$ by convention from Definition 2.4. This polynomial can be decomposed into k pairwise coprime factors

$$P_{[\varphi]}(t) = P_{[\varphi_1]}(t)P_{[\varphi_2]}(t)\dots P_{[\varphi_k]}(t)$$

where $\varphi_j(0) = \pi(B_{0,\lambda_j})$ for each j. Moreover, if Φ is a lifting map of (φ, B_0, B_1), then $\mathrm{Ker}(P_{[\varphi_j(\zeta)]}(\Phi(\zeta)))$ is an invariant subspace of Φ at each value of ζ and

$$\mathbb{C}^n = \bigoplus_{j=1}^{k} \mathrm{Ker}(P_{[\varphi_j(\zeta)]}(\Phi(\zeta))).$$

It means that the local liftability of (φ, B_0, B_1) is equivalent to the simultaneous local liftability of k new data $(\varphi_j, B_{0,\lambda_j}, B_{1,\lambda_j})$.

It is therefore sufficient to consider only the case where B_0 has only one eigenvalue, and in that case, by applying a Mobius transformation $M \mapsto (M - \lambda I)(I - \bar{\lambda} M)^{-1}$, we can suppose B_0 is nilpotent.

3.3 Box product

To compute the derivatives of $\pi \circ \Phi$, we have to linearize the polynomials $\sigma_i(M)$ with $M \in \mathbb{C}^{n,n}$. The tool for this linearization is called *box product*. We do not present this tool here, but say briefly that: $\sigma_i(M)$ will be the $i-$multilinear product of i matrices M. Denote

$$\sigma_i(M) = \frac{1}{i!} M \cdot M \cdot \ldots \cdot M.$$

In the thesis we will omit the $\cdot$ without confusion with matrix multiplication because we hardly use matrix multiplication.

3.4 First analyses of the problem

Suppose a datum $\{\varphi, B_0, B_1\}$ has a local lifting Φ. As above, suppose B_0 is nilpotent. We recall the numbers d_i associated with B_0 which were presented in Definition 2.6. Precisely, denote by (b_{ij}) the entries of B_0 (sometimes we write $(b_{i,j})$ instead if there is a need to separate the two indices), then put

$$\begin{aligned} F_0 &= \{j \colon \ b_{j-1,j} = 0\} \cup \{1\} \\ &= \{b_1 = 1 < b_2 < \ldots < b_s\}. \end{aligned}$$

The number s here is the number of elementary Jordan blocks of B_0.

Next, we arrange the elementary Jordan blocks of B_0 so that $b_{j+1} - b_j$ is increasing with convention that $b_{s+1} = \text{size}(B_0) + 1 = n + 1$. Then

$$d_i = 1 + \sharp \left(F_0 \cap [n + 2 - i..n] \right).$$

Since the datum $\{\varphi, B_0, B_1\}$ is locally liftable, then by the previous chapter,

$$P^{(k)}_{[\varphi(\zeta)]}(0) = \mathcal{O}(\zeta^{d_{n-k}}) \text{ for } 0 \leq k \leq n-1.$$

But $P^{(k)}_{[\varphi(\zeta)]}(0) = k!\sigma_{n-k}(-\Phi(\zeta)) = (-1)^{n-k} k! \sigma_{n-k}(\Phi(\zeta)) = (-1)^{n-k} k! \varphi_{n-k}(\zeta)$. Therefore, for nilpotent B_0, the conditions are of simple form as follows

$$\varphi_k(\zeta) = \mathcal{O}(\zeta^{d_k}).$$

Denote by $\Phi^{(k)}(0) = B_k$ the k-th derivative of Φ at $\zeta = 0$. Of course B_0, B_1 are given, so the other B_k's are to be found. Naturally, the relations between these B_k's are governed by the conditions of φ at $\zeta = 0$, i.e., the values of its derivatives there. So we have to compute these derivatives in terms of Φ, i.e., to compute

$$\left.\frac{d^k}{d\zeta^k}\right|_{\zeta=0} \sigma_m(\Phi(\zeta)).$$

As we know from box product,

$$\sigma_m(\Phi) = \frac{1}{m!}\mathrm{Tr}(\underbrace{\Phi\Box\ldots\Box\Phi}_{m \text{ times}}) = \frac{1}{m!}\underbrace{\Phi\cdot\ldots\cdot\Phi}_{m \text{ times}}.$$

By Leibniz's general rule,

$$\left.\frac{d^k}{d\zeta^k}\right|_{\zeta=0} \sigma_m(\Phi(\zeta)) = \frac{1}{m!}\sum_{\substack{j_1+j_2+\ldots+j_m=k\\ j_i\geq 0}} \frac{k!}{j_1!j_2!\ldots j_m!}\Phi^{(j_1)}(0)\Phi^{(j_2)}(0)\ldots\Phi^{(j_m)}(0) \tag{3.2}$$

$$= \frac{1}{m!}\sum_{\substack{j_1+j_2+\ldots+j_m=k\\ j_i\geq 0}} \frac{k!}{j_1!j_2!\ldots j_m!}B_{j_1}B_{j_2}\ldots B_{j_m}. \tag{3.3}$$

In other words,

$$\varphi_m^{(k)}(0) = \frac{1}{m!}\sum_{\substack{j_1+j_2+\ldots+j_m=k\\ j_i\geq 0}} \frac{k!}{j_1!j_2!\ldots j_m!}B_{j_1}B_{j_2}\ldots B_{j_m}. \tag{3.4}$$

They are equations in B_k for $k \geq 2$. A solution $\{B_k\}_{k\geq 2}$ of these equations does not necessarily form a local holomorphic mapping Φ around $\zeta = 0$. It means we have to treat the convergence problem of the solution. We will discuss it later.

3.5 The cases for B_0

Statements of results and proofs of Problem 3.1 will be presented according to the cases of B_0, more precisely, the Jordan form of B_0. Since B_0 is nilpotent and non-cyclic[1], there are three cases to consider:

[1]The case where B_0 is cyclic was dealt with in the article of Huang, Marcantognini and Young.

(I) $B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(III) $B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(II) $B_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

3.6 Associated linear mapping L_{B_0,B_1}, conditions and a scheme of proof

Let $\{\varphi, B_0, B_1\}$ be a datum of the lifting problem. As we argued before, we can suppose B_0 is nilpotent and in the Jordan form. Denote by d_i the numbers associated with B_0. For example, if

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then $d_1 = d_2 = d_3 = 1$ and $d_4 = 2$.

If φ is locally liftable, then $\varphi(0) = (0, 0, \ldots, 0) \in \mathbb{C}^n$. Here we look at the general case where n is the size of the matrix B_0. But we can only provide proofs for the case where $n = 4$ in the following sections.

We know from the Chapter 2, this condition implies that

$$P^{(k)}_{[\varphi(\zeta)]}(0) = \mathcal{O}(\zeta^{d_{n-k}})$$

for $0 \leq k \leq n-1$. But the LHS is $k!(-1)^{n-k}\varphi_{n-k}(\zeta)$, so we obtain simple conditions

$$\varphi_j(\zeta) = \mathcal{O}(\zeta^{d_j}) \text{ for } 1 \leq j \leq n. \tag{3.5}$$

These conditions give a corollary as follows.

Corollary 3.3. *Suppose $B_0 \in \mathbb{C}^{n,n}$ is nilpotent and $d_1, \ldots, d_n$ its associated numbers. For $M_1, M_2, \ldots, M_k \in \mathbb{C}^{n,n}$ and $d_i \geq k+1$, we always have*

$$\underbrace{B_0 B_0 \ldots B_0}_{i-k \text{ times}} M_1 M_2 \ldots M_k = 0.$$

Recall that we always denote by B_k the k-th derivative $\Phi^{(k)}(0)$ of the lifting map Φ. Thanks to Corollary 3.3, the equations at (3.4) are of the form

$$\varphi_m^{(k+d_m-1)}(0) = \frac{(k+d_m-1)!}{k!} \frac{\overbrace{B_0B_0\dots B_0}^{m-d_m \text{ times}}\overbrace{B_1B_1\dots B_1}^{d_m-1 \text{ times}}B_k}{(m-d_m)!(d_m-1)!} + \dots \tag{3.6}$$

where the dots represent the terms depending only on B_i with $i < k$. This suggests that we should arrange the equations at (3.4) into groups of n equations. Precisely, put

$$\mathfrak{X}_k = \left\{ \frac{\varphi_1^{(k+d_1-1)}(0)}{(k+1)(k+2)\dots(k+d_1-1)}, \frac{\varphi_2^{(k+d_2-1)}(0)}{(k+1)(k+2)\dots(k+d_2-1)}, \dots \right.$$
$$\left. , \frac{\varphi_n^{(k+d_n-1)}(0)}{(k+1)(k+2)\dots(k+d_n-1)} \right\}. \tag{3.7}$$

Then the left hand side of the equations of φ (3.4) imposed by the value of the k-th derivative of the mapping φ, for any $k \geq 2$, is precisely $\mathfrak{X}_k$ for $k \geq 2$ (remark that for $k = 0$, and 1, B_0 and B_1 are given).

We describe the right hand side of those equations. Consider the following linear mapping $L_{B_0,B_1} \colon \mathbb{C}^{n,n} \to \mathbb{C}^n$ where the m-th coordinate of $L_{B_0,B_1}(M)$ for $M \in \mathbb{C}^{n,n}$ is defined by the formula

$$\frac{\overbrace{B_0B_0\dots B_0}^{m-d_m \text{ times}}\overbrace{B_1B_1\dots B_1}^{d_m-1 \text{ times}}M}{(m-d_m)!(d_m-1)!}.$$

We call this mapping L_{B_0,B_1} the *associated linear mapping* with the lifting problem (or with the datum $\{\varphi, B_0, B_1\}$).

Now we look at the $\mathfrak{X}_k$ for $k \geq 2$. It is in fact of the form

$$\mathfrak{X}_k = L_{B_0,B_1}(B_k) + \dots$$

where the dots represent the terms only depending on B_i with $i < k$.

We remark that if $\operatorname{rank}(L_{B_0,B_1}) = n$, i.e., L_{B_0,B_1} is of full rank, then we can find a sequence $\{B_k\}_{k\geq 2}$ which solves the equations where B_k has to be found before B_{k+1}.

However, this does not ensure the convergence of the series $\sum_{k=0}^{\infty} \frac{B_k}{k!}\zeta^k$ which is necessary for Φ being holomorphic locally around $\zeta = 0$. We will discuss of this problem later.

We remark that if $\operatorname{rank}(L_{B_0,B_1}) < n$, this gives a linear relation between the quantities $\mathfrak{X}_k$. Precisely, B_k can be eliminated from some linear combination of the right hand side of (3.4) to yield an equation free of B_k, i.e. only depending on B_0, B_1, ..., B_{k-1}. Remark that we consider $B_2, B_3, \ldots,$ as a series of information where B_{k-1} is known before B_k. When $k = 2$, this gives concrete necessary conditions on φ, and this is exactly how N. Nikolov, P. Pflug and P. J. Thomas obtained necessary conditions, even though they might do it implicitly.

In the case where $\operatorname{rank}(L_{B_0,B_1}) < n$, some quantities of $\mathfrak{X}_k$ do not involve B_k, for example, $\varphi_n^{(k+d_n-1)}(0)$. It is natural to consider the next derivatives, for example, $\varphi_n^{(k+d_n)}(0)$ until it depends on B_k but not B_{k+1}. By this way, we in fact replace L_{B_0,B_1} by a new map L_{B_0,B_1,B_2} where B_2 participates in the definition of L_{B_0,B_1,B_2} but no other B_k for $k \geq 3$. The new map L_{B_0,B_1,B_2} must be of full rank.

But it always remains the problem of convergence of $\sum_{k=0}^{\infty} \frac{B_k}{k!}\zeta^k$. It turns out not to be too difficult: if we can find a solution $\{B_k\}$ of the form where B_k has nonzero entries in the last row only, the convergence is ensured, since the sequence B_k will be generated by a kind of recurrent relation, and we can estimate the norms of its terms.

To do this, we have to find a suitable B_2 so that the restriction of L_{B_0,B_1,B_2} to the subspace of $\mathbb{C}^{n,n}$ composed of matrices with nonzero entries in the last row only is of full rank, i.e., equal to n.

The proof will then be mainly technical and of course, lengthy.

3.7 Main results

Readers are referred to Conclusion.

Conclusion

The following are the main results in the thesis with the same numbering.

Theorem 1.7 Let X be a complex manifold which contains an entire curve, i.e. there exists a non-constant holomorphic curve $f\colon \mathbb{C} \to X$. Then both $\mathbb{C} \times X$ and $\mathbb{C}^* \times X$ are not of $E-$limit type for any hermitian metric E on $\mathbb{C} \times X$ or $\mathbb{C}^* \times X$ respectively.

Proposition 2.11 Let $\varphi \in Hol(\omega, \mathbb{G}_n)$, where ω is a neighborhood of $\alpha \in \mathbb{D}$. Let A_1 be as in (2.1). Then the following assertions are equivalent:

(a) There exists $\omega' \subset \mathbb{D}$ a neighborhood of α and $\Phi \in \mathrm{Hol}(\omega', \Omega_n)$ such that

$$\pi \circ \Phi = \varphi, \Phi(0) = A_1;$$

(b) The map φ verifies

$$\frac{d^k P_{[\varphi(\zeta)]}}{dt^k}(\lambda_j) = O((\zeta - \alpha)^{d_{m_j - k}(B_j)}), \quad 0 \leq k \leq m_j - 1, 1 \leq j \leq s, \quad (3.8)$$

where the d_i are as in Definition 2.6.

(c) There exists $\omega' \subset \mathbb{D}$ a neighborhood of α and $\Phi \in Hol(\omega', \Omega_n)$ such that

$$\pi \circ \Phi = \varphi, \Phi(0) = A_1 \text{ and } \Phi(\zeta) \text{ is cyclic for } \zeta \in \omega' \setminus \{\alpha\}.$$

Theorem 2.20 Let $n \in \mathbb{N}^*$, $n \leq 5$, $A_1, \ldots, A_N \in \mathcal{M}_n$, $\alpha_1, \ldots, \alpha_N \in \mathbb{D}$ and $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{G}_n)$.

Then there exists $\Phi \in \mathrm{Hol}(\mathbb{D}, \Omega_n)$ satisfying $\varphi = \pi \circ \Phi$ and $\Phi(\alpha_j) = A_j$ for $j = 1, \ldots, N$ if and only if φ satisfies the conditions (3.8) for each j, with A_j instead of A_1 and α_j instead of α, for $1 \leq j \leq N$.

Furthermore, the values $\Phi(\zeta)$ may be chosen as cyclic matrices when $\zeta \notin \{\alpha_1, \ldots, \alpha_n\}$.

Theorem 3.7 Suppose $\mathrm{rank}(L_{B_0,B_1}) = 3$. The datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0$,

(ii) $\varphi_1'(0) = \sigma_1(B_1)$, $\varphi_2'(0) = B_0B_1$, $\varphi_3'(0) = \dfrac{B_0B_0B_1}{2}$,

(iii) $\varphi_4'(0) = 0$, $\varphi_4''(0) = \dfrac{B_0B_0B_1B_1}{2} = 0$,

(iv) $\dfrac{\varphi_4'''(0)}{3} - b_{11}\varphi_3''(0) = \dfrac{B_0B_1B_1B_1}{3} - b_{11}B_0B_1B_1$.

Theorem 3.8 Suppose $\mathrm{rank}(L_{B_0,B_1}) = 4$. The datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0$,

(ii) $\varphi_1'(0) = \sigma_1(B_1)$, $\varphi_2'(0) = B_0B_1$, $\varphi_3'(0) = \dfrac{B_0B_0B_1}{2}$,

(iii) $\varphi_4'(0) = 0$, $\varphi_4''(0) = \dfrac{B_0B_0B_1B_1}{2} = 0$.

Theorem 3.9 Suppose $\mathrm{rank}(L_{B_0,B_1}) = 2$. The datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0$,

(ii) $\varphi_1'(0) = \sigma_1(B_1)$, $\varphi_2'(0) = B_0B_1$,

(iii) $\varphi_3'(0) = \varphi_4'(0) = 0$,

(iv) $\varphi_3''(0) = B_0B_1B_1$,

(v) $\varphi_4''(0) = \frac{B_0B_0B_1B_1}{2} = 2\begin{vmatrix} b_{21} & b_{23} \\ b_{41} & b_{43} \end{vmatrix} = 0$,

(vi) $\varphi_4'''(0) = B_0B_1B_1B_1$,

(vii) $\frac{\varphi_3'''(0)}{3} - \alpha\varphi_2''(0) = 2\sigma_3(B_1) - 2\alpha\sigma_2(B_1)$.

Theorem 3.10 Suppose $\operatorname{rank}(L_{B_0,B_1}) = 3$. The datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0$,

(ii) $\varphi_1'(0) = \sigma_1(B_1)$, $\varphi_2'(0) = B_0B_1$,

(iii) $\varphi_3'(0) = \varphi_4'(0) = 0$,

(iv) $\varphi_3''(0) = B_0B_1B_1$,

(v) $\varphi_4''(0) = \frac{B_0B_0B_1B_1}{2} = 2\begin{vmatrix} b_{21} & b_{23} \\ b_{41} & b_{43} \end{vmatrix}$,

(vi) $\varphi_4'''(0) + 3\kappa\varphi_2''(0) = B_0B_1B_1B_1 + 6\kappa\sigma_2(B_1)$.

Theorem 3.11 Suppose $\operatorname{rank}(L_{B_0,B_1}) = 4$. The datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0$,

(ii) $\varphi_1'(0) = \sigma_1(B_1)$, $\varphi_2'(0) = B_0B_1$,

(iii) $\varphi_3'(0) = \varphi_4'(0) = 0$,

(iv) $\varphi_3''(0) = B_0B_1B_1$,

(v) $\varphi_4''(0) = \frac{B_0B_0B_1B_1}{2} = 2\begin{vmatrix} b_{21} & b_{23} \\ b_{41} & b_{43} \end{vmatrix}$.

Theorem 3.12 In the case where $\operatorname{rank}(L_{B_0,B_1}) = 2$ and $\{b_{11}, b_{12}, b_{21}, b_{22}\} \neq \{0\}$ the datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi(0) = \pi(B_0) = (0, 0, 0, 0)$,

(ii) $\varphi'(0) = (\sigma_1(B_1), B_0B_1, 0, 0) = (\sigma_1(B_1), -b_{43}, 0, 0)$,

(iii) $\varphi_3''(0) = B_0B_1B_1$,

(iv) $\varphi_4''(0) = 0$,

(v) $\underbrace{\frac{\varphi_3'''(0)}{3} - 2\sigma_3(B_1)}_{=B_0B_1B_2} = (b_{11} + b_{22})\Big(\underbrace{\varphi_2''(0) - 2\sigma_2(B_1)}_{B_0B_2}\Big)$

(vi) $\underbrace{\frac{\varphi_4^{(4)}(0)}{12} - 2\sigma_4(B_1)}_{=\frac{B_0B_1B_1B_2}{2}} = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}\Big(\underbrace{\varphi_2''(0) - 2\sigma_2(B_1)}_{=B_0B_2}\Big)$.

Theorem 3.13 In the case where $\operatorname{rank}(L_{B_0,B_1}) = 2$ and $\{b_{11}, b_{12}, b_{21}, b_{22}\} = \{0\}$ the datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi(0) = \pi(B_0) = (0, 0, 0, 0)$,

(ii) $\varphi'(0) = (\sigma_1(B_1), B_0B_1, 0, 0) = (\sigma_1(B_1), -b_{43}, 0, 0)$,

(iii) $\varphi_3''(0) = B_0B_1B_1$,

(iv) $\varphi_4''(0) = 0$,

(v) $\dfrac{\varphi_3'''(0)}{3} = \dfrac{\varphi_4^{(4)}(0)}{12} = 0$

(vi) $\varphi_4^{(5)}(0) = 0$.

Theorem 3.14 If $\operatorname{rank}(L_{B_0,B_1}) = 3$, the datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi_1(0) = \varphi_2(0) = \varphi_3(0) = \varphi_4(0) = 0$,

(ii) $\varphi_1'(0) = \sigma_1(B_1)$, $\varphi_2'(0) = B_0B_1 = -b_{43}$,

(iii) $\varphi_3'(0) = \varphi_4'(0) = 0$,

(iv) $\varphi_3''(0) = B_0B_1B_1$, $\varphi_4''(0) = 0$,

(v) $\varphi_4'''(0) = B_0B_1B_1B_1$,

(vi) $\dfrac{\varphi_4^{(4)}(0)}{12} - \kappa\dfrac{\varphi_3'''(0)}{3} - \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \varphi_2''(0) = 2\sigma_4(B_1) - 2\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \sigma_2(B_1)$
$- 2\kappa\sigma_3(B_1) + 2\kappa(b_{11} + b_{22})\sigma_2(B_1)$.

Theorem 3.15 The datum $\{\varphi, B_0, B_1\}$ is locally liftable iff the following are verified

(i) $\varphi(0) = \pi(B_0)$,

(ii) $\varphi'(0) = \mathrm{D}\pi_{B_0}B_1$ where $\mathrm{D}\pi_{B_0}$ is the total derivative of π at B_0,

(iii) $\varphi_3''(0) = B_0B_1B_1$, $\varphi_4''(0) = 0$, $\varphi_4'''(0) = B_0B_1B_1B_1$.

Published articles and preprint of the author

[1] Nikolov, Nikolai, Thomas, Pascal J., and Tran, Duc-Anh. "Lifting maps from the symmetrized polydisc in small dimensions". *Complex Anal. Oper. Theory* 10.5 (2016), pp. 921–941.

[2] Tran, Duc-Anh. "Lifting map problem from the symmetrized polydisc $\mathbb{G}_4$ with given first derivative". Preprint.

[3] Tran, Duc-Anh. "On the non-existence of limit E-Brody curves". *Acta Math. Vietnam.* 41.4 (2016), pp. 711–714.